

# ETALE HOMOTOPY AND SUMS-OF-SQUARES FORMULAS

DANIEL DUGGER AND DANIEL C. ISAKSEN

ABSTRACT. This paper uses a relative of  $BP$ -cohomology to prove a theorem in characteristic  $p$  algebra. Specifically, we obtain some new necessary conditions for the existence of sums-of-squares formulas over fields of characteristic  $p > 2$ . These conditions were previously known in characteristic zero by results of Davis. Our proof uses a generalized étale cohomology theory called étale  $BP2$ .

## CONTENTS

1. Introduction	1
2. Sums-of-squares and complex-oriented theories	4
3. The main results	8
4. Introduction to the rest of the paper	11
5. Generalized cohomology of pro-spaces	12
6. Étale cohomology and $\mathcal{SP}$ -linear schemes	15
7. Étale realizations	17
References	22

## 1. INTRODUCTION

Let  $F$  be a field. A **sums-of-squares formula** of type  $[r, s, n]$  is an identity of the form

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2$$

where each  $z_i$  is a bilinear expression in the  $x$ 's and  $y$ 's with coefficients in  $F$ . The identity occurs in the polynomial ring  $F[x_1, \dots, x_r, y_1, \dots, y_s]$ . A classical problem, raised by Hurwitz in 1898 [H1], is to determine which triples  $[r, s, n]$  can admit a sums-of-squares formula (with the answer possibly depending on the field  $F$ ). In this paper we show how étale homotopy theory and classical algebraic topology combine to yield necessary conditions for the existence of such a formula, applicable even to fields of characteristic  $p > 2$ . Specifically, we use the generalized cohomology theory  $BP2$  to obtain some conditions previously unknown in the positive characteristic case.

Hurwitz introduced the sums-of-squares problem after he proved that a formula of type  $[n, n, n]$  can exist only if  $n$  is 1, 2, 4, or 8 [H1]. This is his classical theorem on the existence of composition algebras. Technically speaking, Hurwitz proved this theorem only when  $F = \mathbb{R}$ , but his methods work for arbitrary  $F$  not of characteristic 2. (Note that the whole sums-of-squares problem is trivial in characteristic 2, where formulas of all possible types exist). Later, Hurwitz and Radon [H2, R] completely analyzed the case of formulas of type  $[r, n, n]$ .

When  $F = \mathbb{R}$ , topological methods have provided a series of necessary conditions for a formula of type  $[r, s, n]$  to exist. The earliest papers in this vein are by Hopf [H] and Stiefel [St], who showed that if a formula of type  $[r, s, n]$  exists over  $\mathbb{R}$  then there is a map of spaces

$$\phi: \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \rightarrow \mathbb{R}P^{n-1}$$

which restricts to linear inclusions on  $\mathbb{R}P^{r-1} \times \{*\}$  and  $\{*\} \times \mathbb{R}P^{s-1}$  (for any choice of basepoints). Such a  $\phi$  is called an **axial map** of projective spaces. Hopf used singular  $\mathbb{Z}/2$ -cohomology to obtain topological obstructions to the existence of such a map.

Topologists have maintained an interest in the existence of axial maps because of its relation to the immersion problem for  $\mathbb{R}P^k$ . Generalized cohomology theories supply more and more stringent conditions on the existence of axial maps, developed in papers such as [As], [D1], [D2], and [BDM]. The paper [A] also dealt with the immersion problem—and, implicitly, the existence of sums-of-squares formulas—but not through the axial map perspective.

These topological methods give conditions on the existence of sums-of-squares formulas over  $\mathbb{R}$ . As explained in [L, Sec. 2], these conditions also apply to any field of characteristic zero. The present paper continues the sequence [DI1, DI2] in which we prove that these results also apply to characteristic  $p$  fields. The topological methods are replaced by methods of motivic homotopy theory. The paper [DI1] used mod 2 motivic cohomology, and [DI2] used algebraic  $K$ -theory. The present paper uses étale  $BP$ -theory.

For additional background on the sums-of-squares problem over characteristic zero fields, the reader may consult [L] and [Sh].

We now describe the methods of the paper in more detail. Recall that we have fixed a field  $F$ , and let  $Q_{n-1} \hookrightarrow \mathbb{P}^n$  be the quadric defined over  $F$  by the equation  $x_0^2 + \cdots + x_n^2 = 0$ . Write  $DQ_n = \mathbb{P}^n - Q_{n-1}$  (the subscript on a scheme indicates its dimension). The scheme  $DQ_n$  is in some sense a motivic analogue of  $\mathbb{R}P^n$ . One facet of this analogy is that when  $F = \mathbb{C}$ , the space  $\mathbb{R}P^n$  is homotopy equivalent to the topological space  $DQ_n(\mathbb{C})$  of  $\mathbb{C}$ -valued points [Lw]. In [SS] and [DI1] it was observed that if a sums-of-squares formula of type  $[r, s, n]$  exists over  $F$ , then there is an axial map of  $F$ -varieties

$$\phi: DQ_{r-1} \times DQ_{s-1} \rightarrow DQ_{n-1}.$$

Here is a list of the main steps in the paper:

- (1) We prove that if a sums-of-squares formula of type  $[r, s, n]$  exists over  $\mathbb{R}$  then a diagram of the form

$$\begin{array}{ccc} \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} & \xrightarrow{j \times j} & \mathbb{R}P^\infty \times \mathbb{R}P^\infty \\ \downarrow & & \downarrow \mu \\ \mathbb{R}P^{n-1} & \xrightarrow{j} & \mathbb{R}P^\infty \end{array}$$

exists in the homotopy category of topological spaces. Here  $j$  denotes the obvious inclusions, and  $\mu$  is the usual multiplication map. Consequently, if  $E^*$  is a complex-oriented cohomology theory then one has  $(X \otimes 1 - 1 \otimes X)^e = 0$  in

- $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ , where  $e = \lfloor \frac{n+1}{2} \rfloor$ . We use  $X$  to denote the complex-orientation class in  $E^2(\mathbb{C}P^\infty)$ , as well as its restriction to  $E^2(\mathbb{R}P^\infty)$  or to  $E^2(\mathbb{R}P^m)$ .
- (2) We next prove that if a sums-of-squares formula of type  $[r, s, n]$  exists over any field  $F$  not of characteristic 2, then there exists a similar diagram in the homotopy category  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}\mathcal{S}et)$  of pro-simplicial sets and  $\mathbb{Z}/2$ -cohomological equivalences. This uses étale homotopy theory and the map  $\phi$  mentioned above. The key step is an analysis of the étale homotopy type of  $DQ_n$ .
  - (3) Suppose  $E$  is a complex-oriented cohomology theory that is  $k$ -connected for some  $k$ , and is such that each coefficient group  $E^q$  is a finite 2-group. As a formal consequence of the diagram from (2), one again obtains the identity  $(X \otimes 1 - 1 \otimes X)^e = 0$  in  $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ .
  - (4) Putting everything together, the existence of a sums-of-squares formula of type  $[r, s, n]$  over an arbitrary field (not of characteristic 2) implies that a certain algebraic expression vanishes in  $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ , for any cohomology theory  $E$  of the type described in (3). In particular, one may take  $E^*(X) = BP2^*(X; \mathbb{Z}/2^N)$  for large  $N$ , in which case one can use the  $BP2$  calculations done by Davis [D2]. This yields a new necessary condition for the existence of sums-of-squares formulas over fields of positive characteristic, as stated in the following theorem. This result is proved at the end of Section 3.

**Theorem 1.1.** *Suppose a sums-of-squares formula of type  $[2a + 1, 2b + 1, 2m - 1]$  exists over a field  $F$ , where  $\max\{a, b\} < m \leq a + b$  and  $\mathrm{char}(F) \neq 2$ . Then the vector*

$$[(-1)^a \binom{m}{a}, (-1)^{a-1} \binom{m}{a-1}, \dots, (-1)^{m-b} \binom{m}{m-b}]$$

*is in the  $\mathbb{Z}$ -linear span of the relations listed in Theorem 2.7 below.*

Unfortunately the relations mentioned in the above theorem (which are here interpreted as vectors of integers) are too awkward to list in the introduction. However, we give the following concrete application which is discussed in more detail in Examples 2.9 and 2.11. Consider whether a sums-of-squares formula of type  $[11, 15, 17]$  exists over a field of characteristic not equal to 2. The necessary conditions obtained in [DI1] and [DI2] are satisfied, which means that those results are inconclusive. According to the above theorem, however, if such a formula exists, then the vector  $[-126, 126, -84, 36]$  belongs to the  $\mathbb{Z}$ -linear span of

$$[2, 0, 0, 0], \quad [0, 2, 2, 2], \quad [0, 0, 4, 4], \quad \text{and} \quad [0, 0, 0, 2].$$

A quick calculation shows that a formula of type  $[11, 15, 17]$  therefore does not exist.

**Remark 1.2.** The paper [BDM] applied the cohomology theory  $tmf$  to the axial map problem. Its results can be interpreted as giving further restrictions on the possible dimensions of sums-of-squares formulas over  $\mathbb{R}$ . According to the authors, those restrictions actually improve on the  $BP2$  restrictions by arbitrarily large amounts as the dimensions get big. We have not explored whether these  $tmf$  results also extend to characteristic  $p$  fields (the arguments of the present paper do not quite apply because  $tmf$  is not complex-oriented); so this remains an interesting open question.

**1.3. Organization.** The paper is organized as follows. Section 2 is expository. It describes the approach of Astey and Davis to sums-of-squares formulas over  $\mathbb{R}$ , which is important background for understanding our arguments. Section 3

introduces the basic machinery of our arguments and outlines the proof of Theorem 1.1. The proofs of two key steps, Propositions 3.2 and 3.4, are postponed until later.

The remainder of the paper is dedicated to the task of proving these two propositions. Section 4 provides an overview of the technical issues. The last three sections fill in the details, as well as provide more detailed background on the ideas in Section 3.

## 2. SUMS-OF-SQUARES AND COMPLEX-ORIENTED THEORIES

In this section we describe how complex-oriented cohomology theories can be applied to the sums-of-squares problem over  $\mathbb{R}$ . This technique was developed by Astey [As] and later used by Davis [D2]. The specific results we need seem not to be in the literature, so we briefly develop them here.

Suppose that  $E$  is a complex-oriented cohomology theory [Ad, II.2] equipped with an orientation class  $X$  in  $\tilde{E}^2(\mathbb{C}P^\infty)$ . Recall that complex-oriented cohomology theories are represented by spectra with a homotopy associative and homotopy commutative multiplication.

We will abuse notation and also write  $X$  for the image of the orientation class in  $\tilde{E}^2(\mathbb{R}P^\infty)$  under the standard inclusion  $\mathbb{R}P^\infty \hookrightarrow \mathbb{C}P^\infty$ . Even further, we write  $X$  also for the image in  $\tilde{E}^2(\mathbb{R}P^n)$  under the standard inclusion  $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$ . The resulting classes  $X \otimes 1$  and  $1 \otimes X$  in  $E^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$  or  $E^*(\mathbb{R}P^m \times \mathbb{R}P^n)$  will be denoted  $X_1$  and  $X_2$ .

**Lemma 2.1.** *Let  $E$  be a complex-oriented cohomology theory. Then  $X^e = 0$  in  $E^*(\mathbb{R}P^{n-1})$ , where  $e = \lfloor \frac{n+1}{2} \rfloor$ .*

*Proof.* Recall that any complex-oriented cohomology theory  $E$  comes equipped with a (multiplicative) structure map of associated spectra  $MU \rightarrow E$  [Ad, II.4.6]. By naturality, we only need to consider the case  $E = MU$ . Consider the Atiyah-Hirzebruch spectral sequence [AH]

$$E_2^{p,q} = H^p(\mathbb{R}P^{n-1}; MU^q) \Rightarrow MU^{p+q}(X).$$

The  $E_2$ -term vanishes unless  $0 \leq p \leq n-1$  and  $q \leq 0$ . It follows that  $MU^i(\mathbb{R}P^{n-1})$  is zero for  $i > n-1$ .

Since the degree of  $X^e$  is at least  $n$ , we conclude that  $X^e = 0$ .  $\square$

The above proof explains the choice of exponent  $e = \lfloor \frac{n+1}{2} \rfloor$ . It is the smallest integer such that  $2e \geq n$ .

**Lemma 2.2.** *Let  $\mu : \mathbb{R}P^\infty \times \mathbb{R}P^\infty \rightarrow \mathbb{R}P^\infty$  be the usual map that represents tensor product of line bundles, and let  $E$  be a complex-oriented cohomology theory. Then  $\mu^*(X) = u(X_1 - X_2)$ , where  $u$  is a unit in  $E^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$ .*

*Proof.* As in the proof of Lemma 2.1, it suffices to consider the case  $E = MU$ , which is precisely [As, Prop. 3.6].  $\square$

**Definition 2.3.** *A map  $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \rightarrow \mathbb{R}P^{n-1}$  is **axial** if there exist base-points of  $\mathbb{R}P^{r-1}$  and  $\mathbb{R}P^{s-1}$  such that the restricted maps  $\mathbb{R}P^{r-1} \times * \rightarrow \mathbb{R}P^{n-1}$  and  $* \times \mathbb{R}P^{s-1} \rightarrow \mathbb{R}P^{n-1}$  take the generator in  $H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2)$  to the generators of  $H^1(\mathbb{R}P^{r-1}; \mathbb{Z}/2)$  and  $H^1(\mathbb{R}P^{s-1}; \mathbb{Z}/2)$ , respectively.*

**Proposition 2.4.** *Suppose that  $f: \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \rightarrow \mathbb{R}P^{n-1}$  is an axial map, and let  $E$  be a complex-oriented cohomology theory. Then  $(X_1 - X_2)^e = 0$  in the ring  $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ , where  $e = \lfloor \frac{n+1}{2} \rfloor$ .*

*Proof.* We first claim that there is a homotopy-commutative diagram

$$\begin{array}{ccc} \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} & \xrightarrow{j \times j} & \mathbb{R}P^\infty \times \mathbb{R}P^\infty \\ f \downarrow & & \downarrow \mu \\ \mathbb{R}P^{n-1} & \xrightarrow{j} & \mathbb{R}P^\infty. \end{array}$$

Using that  $\mathbb{R}P^\infty$  represents  $H^1(-; \mathbb{Z}/2)$ , the homotopy-commutativity of this diagram is equivalent to the statement that

$$f^*: H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2) \rightarrow H^1(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}; \mathbb{Z}/2)$$

sends the generator  $x$  to  $x \otimes 1 + 1 \otimes x$ . This is immediate from the fact that  $f$  is an axial map.

Now apply the cohomology theory  $E$  to the above diagram. Together with Lemma 2.2, this shows that  $f^*: E^2(\mathbb{R}P^{n-1}) \rightarrow E^2(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$  takes  $X$  to  $u(X_1 - X_2)$ , where  $u$  is a unit in  $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ .

We also know that  $X^e = 0$  in  $E^*(\mathbb{R}P^{n-1})$  by Lemma 2.1. It follows that  $(X_1 - X_2)^e = 0$  in  $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ .  $\square$

Now suppose that there is a sums-of-squares formula of type  $[r, s, n]$  over  $\mathbb{R}$ . This yields a bilinear map  $f: \mathbb{R}^r \times \mathbb{R}^s \rightarrow \mathbb{R}^n$  given by

$$(x_1, \dots, x_r; y_1, \dots, y_s) \mapsto (z_1, \dots, z_n),$$

using the notation from the first paragraph of the introduction. This map  $f$  has the property that  $\|f(x, y)\|^2 = \|x\|^2 \|y\|^2$  for all  $x$  and  $y$ . In particular,  $f(x, y) = 0$  only if  $x = 0$  or  $y = 0$ . So  $f$  induces a map  $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \rightarrow \mathbb{R}P^{n-1}$ , which is readily seen to be an axial map. This argument is due to Hopf [H]. Proposition 2.4 therefore gives the following consequence.

**Corollary 2.5.** *Suppose that a sums-of-squares formula of type  $[r, s, n]$  exists over  $\mathbb{R}$ . Then for any complex-oriented cohomology theory  $E$ , one has  $(X_1 - X_2)^e = 0$  in the ring  $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ , where  $e = \lfloor \frac{n+1}{2} \rfloor$ .*

For theories  $E$  where one can do the relevant computations, the above corollary leads to necessary conditions for a formula of type  $[r, s, n]$  to exist over  $\mathbb{R}$ . The obvious choice is to take  $E = MU$ . However, as Astey [As] points out, the groups  $MU^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$  are very large—especially compared to the subgroup generated by monomials in  $X_1$  and  $X_2$ , which is what we are really interested in because  $(X_1 - X_2)^e$  is a linear combination of such monomials. So it's preferable to replace  $MU$  with something smaller but that still carries almost as much of the relevant information. Astey [As, Sec. 5] used the Landweber-exact theory

$$X \mapsto MU^*(X) \otimes MU^*/(v_2, v_4, v_5, v_6, \dots)[v_3^{-1}] \otimes \mathbb{Z}_{(2)},$$

for a certain choice of generators  $v_1, v_2, \dots$  of  $MU^*$ . Later Davis [D2] superseded these results using the theory  $BP2$ . We will describe Davis's work next.

**2.6. Computations for  $BP2$ .** Recall that for any prime  $p$  there is an associated Brown-Peterson cohomology theory  $BP$  [Ad, II.16]. The coefficients are  $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$  with  $\deg v_i = -2(p^i - 1)$ . Here  $\mathbb{Z}_{(p)}$  is the integers localized at  $p$ . Whenever  $BP$  is used in the present paper we will have  $p = 2$ . The cohomology theory  $BP2$  is constructed from  $BP$  by killing the elements  $v_3, v_4, \dots$ . There are at least two ways of constructing  $BP2$ : the methods of Baas-Sullivan [B], or using that  $BP$  is a ring spectrum in a modern category of spectra [EKMM, HSS], in which case  $BP2$  can be constructed as a  $BP$ -module spectrum by iterative cofiber sequences. So  $BP2^* = \mathbb{Z}_{(2)}[v_1, v_2]$ . The fact that  $BP2$  has the structure of a homotopy associative, homotopy commutative ring spectrum is classical; but in fact it is actually an  $A_\infty$ -spectrum by [An, G]. Note that  $BP2$  is complex-oriented via the composition of the multiplicative maps  $MU \rightarrow MU_{(p)} \rightarrow BP \rightarrow BP2$ .

Davis [D2] was able to compute the groups  $BP2^{2m}(\mathbb{R}P^{2a} \times \mathbb{R}P^{2b})$  completely for  $m$  in a range. The following result is [D2, Thm. 1.4]:

**Theorem 2.7** (Davis). *Assume that  $\max\{a, b\} < m \leq a + b$ ; let  $d = a + b - m$ . Then  $BP2^{2m}(\mathbb{R}P^{2a} \times \mathbb{R}P^{2b})$  is the abelian group generated by the  $d + 1$  elements*

$$w_0 = X_1^a X_2^{m-a}, w_1 = X_1^{a-1} X_2^{m-a+1}, \dots, w_i = X_1^{a-i} X_2^{m-a+i}, \dots, w_d = X_1^{m-b} X_2^b$$

*subject to the  $d + 1$  relations:*

(a)  $2w_0 = 0$ .

(b)  $2^{\lfloor \frac{i+2}{2} \rfloor} \left[ \binom{c}{c} w_i + \binom{c+1}{c} w_{i+1} + \binom{c+2}{c} w_{i+2} + \dots + \binom{c+d-i}{c} w_d \right] = 0$  for  $c = \lfloor \frac{i-1}{2} \rfloor$  and  $1 \leq i < d + 1 - \lfloor \frac{d+1}{3} \rfloor$ .

(c)  $2^{d+1-i} w_i = 0$  if  $d + 1 - \lfloor \frac{d+1}{3} \rfloor \leq i \leq d$ .

To better understand the above relations, it is helpful to explore some examples.

**Example 2.8.** Consider  $BP2^{22}(\mathbb{R}P^{20} \times \mathbb{R}P^{20})$ . Then  $a = b = 10$ ,  $m = 11$ , and  $d = 9$ . The group is the quotient of  $\mathbb{Z}^{10}$  by the rowspace of the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 \\ 0 & 0 & 0 & 0 & 8 & 16 & 24 & 32 & 40 & 48 \\ 0 & 0 & 0 & 0 & 0 & 8 & 24 & 48 & 80 & 120 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 48 & 96 & 160 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The general pattern can be described easily. Corresponding to relation (a), the first row always consists of a 2 followed by zeros. Corresponding to the relations in (c), the bottom-right portion of the matrix is always diagonal, with consecutive powers of 2 ascending on the diagonal from the bottom-right corner; there are precisely  $\lfloor \frac{d+1}{3} \rfloor$  of these diagonal entries. Corresponding to the relations in (b), the rest of

the matrix can be filled in as follows. Consider the sequences

$$\begin{aligned} &1, 1, 1, \dots \\ &\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \dots \\ &\binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \dots \\ &\binom{3}{3}, \binom{4}{3}, \binom{5}{3}, \dots \end{aligned}$$

and so forth. Each of these sequences appears in two rows of the matrix (given enough room), multiplied by consecutive powers of 2, and shifted so that the first entry appears on the diagonal.

**Example 2.9.** Consider the group  $BP2^{18}(\mathbb{R}P^{10} \times \mathbb{R}P^{14})$ . Here  $a = 5$ ,  $b = 7$ , and  $m = 9$ , so  $d = 3$  and  $\lfloor \frac{d+1}{3} \rfloor = 1$ . The group is the quotient of  $\mathbb{Z}^4$  by the rowspace of

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

**Corollary 2.10.** *Suppose that there exists a sums-of-squares formula over  $\mathbb{R}$  of type  $[2a + 1, 2b + 1, 2m - 1]$ , where  $\max\{a, b\} < m \leq a + b$ . Then the vector*

$$[(-1)^a \binom{m}{a}, (-1)^{a-1} \binom{m}{a-1}, \dots, (-1)^{m-b} \binom{m}{m-b}]$$

*is in the  $\mathbb{Z}$ -linear span of the relations listed in Theorem 2.7 (where we interpret the relations as vectors of coefficients).*

*Proof.* From Corollary 2.5,  $(X_1 - X_2)^m = 0$  in  $BP2^{2m}(\mathbb{R}P^{2a} \times \mathbb{R}P^{2b})$  because  $m = \lfloor \frac{2m-1+1}{2} \rfloor$ . Expand this expression to obtain

$$\sum_{k=0}^m (-1)^k \binom{m}{k} X_1^k X_2^{m-k} = 0.$$

By Lemma 2.1,  $X_1^k X_2^{m-k} = 0$  unless  $m - b \leq k \leq a$ . So we can simplify the above equation to

$$\sum_{k=m-b}^a (-1)^k \binom{m}{k} w_{a-k} = 0;$$

here we are also using that  $w_{a-k} = X_1^k X_2^{m-k}$  if  $m - b \leq k \leq a$ .  $\square$

**Example 2.11.** Consider the question of whether a sums-of-squares formula of type  $[11, 15, 17]$  exists over  $\mathbb{R}$ . Apply Corollary 2.10 with  $a = 5$ ,  $b = 7$ , and  $m = 9$ . According to Example 2.9, we need to check whether the vector  $[-\binom{9}{5}, \binom{9}{4}, -\binom{9}{3}, \binom{9}{2}]$  is in the  $\mathbb{Z}$ -linear span of the vectors

$$[2, 0, 0, 0], \quad [0, 2, 2, 2], \quad [0, 0, 4, 4], \quad \text{and} \quad [0, 0, 0, 2].$$

One readily checks that it's *not* in the span, so formulas of type  $[11, 15, 17]$  do not exist over  $\mathbb{R}$ . Note that this trivially implies that a sums-of-squares formula of type  $[11, 15, k]$  does not exist for any  $k \leq 17$ .

By comparison, the Hopf-Stiefel condition [H, St] only shows that formulas of type  $[11, 15, 15]$  do not exist, while the Atiyah condition [A] only shows that formulas of type  $[11, 15, 16]$  (or equivalently,  $[15, 11, 16]$ ) do not exist.

### 3. THE MAIN RESULTS

This section gives the main results of the paper. We begin by summarizing the technical machinery needed.

**3.1. Pro-spaces.** Let **pro- $s\mathbf{Set}$**  denote the category of pro-simplicial sets. Recall that an object of this category consists of a small, cofiltered category  $I$  together with a functor  $X: I \rightarrow s\mathbf{Set}$ . If  $Y: J \rightarrow s\mathbf{Set}$  is another such diagram, then the morphisms from  $X$  to  $Y$  are defined to be

$$\mathrm{Hom}_{\mathrm{pro}}(X, Y) = \lim_j \mathrm{colim}_i \mathrm{Hom}_{s\mathbf{Set}}(X_i, Y_j).$$

Objects of  $s\mathbf{Set}$  will be called ‘spaces’, and so objects of **pro- $s\mathbf{Set}$**  are ‘pro-spaces’. Traditional references for pro-categories include [AM] and [SGA4, Sec. I.8], but for our purposes [I1] is more relevant.

If  $X$  is a pro-space and  $M$  is an abelian group, then the **singular cohomology**  $H^*(X; M)$  of  $X$  with coefficients in  $M$  is defined to be  $\mathrm{colim}_s H^*(X_s; M)$  [AM, 2.2]. Since  $X$  is a cofiltered system and cohomology is contravariant, the colimit in the definition of  $H^*(X; M)$  is a filtered colimit.

A **constant pro-space** is one indexed by the trivial category with one object and an identity map. There is a “constant” functor  $c: s\mathbf{Set} \rightarrow \mathrm{pro-}s\mathbf{Set}$  that takes a simplicial set  $X$  to the constant pro-space with value  $X$ . Note that for any space  $X$  and any abelian group  $M$ , the cohomology rings  $H^*(X; M)$  and  $H^*(cX; M)$  are naturally isomorphic.

We will frequently abuse notation in the following way. If  $X$  is a topological space, we write  $cX$  for the constant functor applied to the singular simplicial set of  $X$ .

There are several useful model category structures on pro-spaces. The one that is relevant for us is the  $\mathbb{Z}/2$ -cohomological model structure [I3]. A map of pro-spaces  $X \rightarrow Y$  is called a  **$\mathbb{Z}/2$ -cohomological equivalence** if the induced map  $H^*(Y; \mathbb{Z}/2) \rightarrow H^*(X; \mathbb{Z}/2)$  is an isomorphism. The presence of the model structure establishes the existence of the associated homotopy category  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro-}s\mathbf{Set})$  obtained by formally inverting the class of  $\mathbb{Z}/2$ -cohomological equivalences.

We will need one additional property of the  $\mathbb{Z}/2$ -cohomological homotopy theory of pro-spaces. The proof of the following result will be postponed until Section 5.

**Proposition 3.2.** *For any spectrum  $E$ , there is a naturally defined functor*

$$(E_2^\wedge)^*: \mathrm{pro-}s\mathbf{Set} \rightarrow (\text{graded abelian groups})$$

*called the **2-completion of  $E$** , satisfying the following properties:*

- (i)  $(E_2^\wedge)^*$  takes  $\mathbb{Z}/2$ -cohomological equivalences to isomorphisms;
- (ii) There is a natural transformation  $\eta: E^*(X) \rightarrow (E_2^\wedge)^*(cX)$  for all spaces  $X$ ;
- (iii) If  $E$  is  $k$ -connected for some  $k$  and each coefficient group  $E^q$  is a finite 2-group, then  $\eta$  is a natural isomorphism.

**3.3. The main argument.** The following proposition is the central technical ingredient in the proof of Theorem 1.1, our main result about sums-of-squares formulas. Its proof will be postponed until the end of Section 7.



**Proposition 3.4.** *Let  $F$  be a field of characteristic not equal to 2, and suppose a sums-of-squares formula of type  $[r, s, n]$  exists over  $F$ . Then there exists a commutative diagram in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}\mathbf{sSet})$  of the form*

$$\begin{array}{ccc} c(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}) & \xrightarrow{c(j \times j)} & c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) \\ \downarrow & & \downarrow c(\mu) \\ c\mathbb{R}P^{n-1} & \xrightarrow{c(j)} & c\mathbb{R}P^\infty. \end{array}$$

Here each  $j$  is the standard inclusion, and  $\mu: \mathbb{R}P^\infty \times \mathbb{R}P^\infty \rightarrow \mathbb{R}P^\infty$  is the map classifying tensor product of line bundles.

**Remark 3.5.** The reader should compare the above diagram to the one from the proof of Proposition 2.4. We do not claim that the left vertical map is induced by a map of ordinary spaces. It does exist in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}\mathbf{sSet})$ , but it is not necessarily in the image of the constant functor  $c$  because it is constructed using a homotopy-inverse in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}\mathbf{sSet})$ ; see the second diagram in the proof of Proposition 3.4 given in Section 7.

We will give the proof of the above proposition in subsequent sections; this is where étale homotopy theory enters the picture. For the moment, however, we will just accept the proposition and consider its consequences.

**Proposition 3.6.** *Let  $E$  be a complex-oriented cohomology theory that is  $k$ -connected for some  $k$ . Suppose that a sums-of-squares formula of type  $[r, s, n]$  exists over a field  $F$  of characteristic not equal to 2. Then  $(X_1 - X_2)^e = 0$  in  $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ , where  $e = \lfloor \frac{n+1}{2} \rfloor$ , provided either of the following two conditions is satisfied:*

- (a) *Each  $E^q$  is a finite 2-group;*
- (b) *The kernel and cokernel of  $E^q \xrightarrow{2} E^q$  are finite-dimensional  $\mathbb{Z}/2$ -vector spaces for each  $q$ , and both of  $r$  and  $s$  are odd.*

Several small variations of the above result are possible. For instance, if one assumes that  $E^*$  is concentrated in even degrees then one can prove (b) assuming only that one of  $r$  and  $s$  is odd.

*Proof.* In case (a) we begin by applying the functor  $(E_2^\wedge)^*$  to the diagram from Proposition 3.4 to obtain a diagram of graded abelian groups. We also have the diagram

$$\begin{array}{ccc} \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} & \xrightarrow{j \times j} & \mathbb{R}P^\infty \times \mathbb{R}P^\infty \\ & & \downarrow \mu \\ \mathbb{R}P^{n-1} & \xrightarrow{j} & \mathbb{R}P^\infty \end{array}$$

of spaces, to which we apply the functor  $E^*$ . The natural transformation  $\eta$  from Proposition 3.2 gives us a commutative diagram

$$\begin{array}{ccccc}
 & & (E_2^\wedge)^*c(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}) & \longleftarrow & (E_2^\wedge)^*c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) \\
 & \nearrow & \uparrow & & \nearrow \\
 E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}) & \longleftarrow & E^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) & & \\
 & \downarrow & \uparrow & & \downarrow \\
 & (E_2^\wedge)^*(c\mathbb{R}P^{n-1}) & \longleftarrow & (E_2^\wedge)^*(c\mathbb{R}P^\infty) & \\
 \nearrow & & \uparrow & & \nearrow \\
 E^*(\mathbb{R}P^{n-1}) & \longleftarrow & E^*(\mathbb{R}P^\infty) & & 
 \end{array}$$

in which all four diagonal maps are isomorphisms. Lemma 2.1 shows that the element  $X^e$  of  $E^*(\mathbb{R}P^\infty)$  maps to zero in  $(E_2^\wedge)^*c(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ . Because  $\eta$  is an isomorphism,  $X^e$  also maps to zero in  $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ . On the other hand, Lemma 2.2 says that  $X^e$  maps to a unit multiple of  $(X_1 - X_2)^e$  in  $E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ . We have computed the image of  $X^e$  in two different ways, and we conclude that  $(X_1 - X_2)^e = 0$ .

Case (b) requires more work. Recall that the reduced  $E$ -cohomology  $\tilde{E}^*(Z)$  of a pointed space  $Z$  is defined to be the kernel of the map  $E^*(Z) \rightarrow E^*$  induced by  $*$   $\rightarrow$   $Z$ . By the definition of  $X_1$  and  $X_2$  in terms of the orientation class of the complex-oriented theory  $E$ , we know that  $(X_1 - X_2)^e$  belongs to  $\tilde{E}^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ . Therefore, we are allowed to compute in reduced  $E$ -cohomology instead of unreduced  $E$ -cohomology.

We examine the reduced Atiyah-Hirzebruch spectral sequence [AH]

$$E_2^{p,q} = \tilde{H}^p(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}; E^q) \Rightarrow \tilde{E}^{p+q}(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}).$$

If  $r$  and  $s$  are odd, then the singular homology groups  $\tilde{H}_*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}; \mathbb{Z})$  are all finite-dimensional  $\mathbb{Z}/2$ -vector spaces and vanish above dimension  $r + s - 2$ . Our assumptions on  $E^*$  guarantee that  $\text{Ext}_{\mathbb{Z}}(V, E^q)$  and  $\text{Hom}_{\mathbb{Z}}(V, E^q)$  are finite-dimensional  $\mathbb{Z}/2$ -vector spaces for any finite-dimensional  $\mathbb{Z}/2$ -vector space  $V$ . The universal coefficient theorem shows that each group  $E_2^{p,q}$  in our spectral sequence is also a finite-dimensional  $\mathbb{Z}/2$ -vector space, and that these groups vanish when  $p > r + s - 1$ . It follows immediately that  $\tilde{E}^k(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$  is a finite 2-group for all  $k$ . Thus, there exists a sufficiently large  $N$  such that  $2^N$  kills  $\tilde{E}^{2e}(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ .

Consider the cohomology theory  $E^*(-; \mathbb{Z}/2^N)$ . As long as  $N \geq 3$ , the spectrum representing this theory inherits a homotopy associative and commutative multiplication from  $E$  [O]. So the theory is complex-oriented. Moreover, the theory satisfies the hypotheses of case (a). So we have that  $(X_1 - X_2)^e = 0$  in  $\tilde{E}^{2e}(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}; \mathbb{Z}/2^N)$ . But consider the following three terms of a long exact sequence

$$\tilde{E}^{2e}(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}) \xrightarrow{2^N} \tilde{E}^{2e}(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}) \rightarrow \tilde{E}^{2e}(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}; \mathbb{Z}/2^N).$$

The first map is zero by our choice of  $N$ , so the second map is a monomorphism. Therefore,  $(X_1 - X_2)^e = 0$  in  $\tilde{E}^{2e}(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})$ .  $\square$

Proposition 3.6 lets one adapt many known results in characteristic zero to characteristic  $p > 2$ , simply by varying the choice of  $E$ . The cohomology theory  $MU/(v_2, v_4, v_5, v_6, \dots)[v_3^{-1}] \otimes \mathbb{Z}_{(2)}$  used by Astey [As] can serve as one such choice, but its results are eclipsed by those of  $BP2$ .

**Theorem 3.7.** *Let  $F$  be a field of characteristic not equal to 2. Suppose that a sums-of-squares formula of type  $[2a + 1, 2b + 1, 2m - 1]$  exists over  $F$ , where  $\max\{a, b\} < m \leq a + b$ . Then the vector*

$$\left[ (-1)^a \binom{m}{a}, (-1)^{a-1} \binom{m}{a-1}, \dots, (-1)^{m-b} \binom{m}{m-b} \right]$$

*is in the  $\mathbb{Z}$ -linear span of the relations listed in Theorem 2.7.*

*Proof.* Apply Proposition 3.6 with  $E = BP2$ . Note that  $BP2^*$  is  $\mathbb{Z}_{(2)}[v_1, v_2]$ , where  $v_1$  and  $v_2$  are classes in dimensions  $-2$  and  $-6$  respectively, and  $\mathbb{Z}_{(2)}$  is the integers localized at 2. It is immediate that  $BP2$  satisfies condition (b) of Proposition 3.6. Then use Davis's computations from Theorem 2.7, just as in the proof of Corollary 2.10.  $\square$

**Remark 3.8.** In this section we have used the  $\mathbb{Z}/2$ -cohomological model structure on pro-simplicial sets. We strongly suspect that the homotopy theory of [Mo] is also a suitable context for the above results.

#### 4. INTRODUCTION TO THE REST OF THE PAPER

The remaining sections of the paper are concerned with the proofs of Propositions 3.2 and 3.4. These are the two uncompleted parts of the main argument given in Section 3.

We will be primarily concerned with the following specific schemes.

**Definition 4.1.** *If  $R$  is a ring, let  $Q_{n-1}$  be the quadric hypersurface in  $\mathbb{P}_R^n$  defined by the equation  $x_0^2 + \dots + x_n^2 = 0$ . Let  $DQ_n$  be the open complement  $\mathbb{P}_R^n - Q_{n-1}$ .*

Usually the ring  $R$  will be clear from context, but in cases where it isn't we will write  $Q_{n-1/R}$  and  $DQ_{n/R}$ .

**Remark 4.2.** For any scheme defined over  $\mathbb{C}$ , let  $X(\mathbb{C})$  denote the set of  $\mathbb{C}$ -valued points of  $X$  equipped with the complex-analytic topology. From [Lw, Lem. 6.3], we know that  $DQ_n(\mathbb{C})$  is homotopy equivalent to  $\mathbb{R}P^n$ . So  $(DQ_m \times DQ_n)(\mathbb{C})$  is homotopy equivalent to  $\mathbb{R}P^m \times \mathbb{R}P^n$ .

Here is a brief outline of the remaining steps in the paper:

- (1) In Section 5 we prove Proposition 3.2. We study generalized cohomology theories and the  $\mathbb{Z}/2$ -cohomological model structure for pro-spaces in more detail than in Section 3.1.
- (2) Section 6 concerns the étale cohomology of the schemes  $DQ_n$  and  $DQ_m \times DQ_n$  (see Examples 6.8 and 6.9). We will show in Theorem 6.6 that the étale  $\mathbb{Z}/2$ -cohomology rings of these schemes do not change under certain kinds of extensions of scalars; in fact, they are isomorphic to the singular  $\mathbb{Z}/2$ -cohomology rings of  $\mathbb{R}P^n$  and  $\mathbb{R}P^m \times \mathbb{R}P^n$ . In order to do this, we first have to extend some standard results about étale cohomology from the class of smooth, projective schemes to the larger class of “ $\mathcal{SP}$ -linear” schemes. These are the schemes that can be built from smooth, projective schemes by iteratively taking complements

of closed inclusions. The ideas in this section are entirely algebro-geometric, not homotopical.

- (3) In Section 7 we will study the étale realizations of  $DQ_n$  and  $DQ_m \times DQ_n$ . We will show that over a separably closed field these étale realizations are  $\mathbb{Z}/2$ -cohomologically equivalent to the constant pro-spaces  $c\mathbb{R}P^n$  and  $c(\mathbb{R}P^m \times \mathbb{R}P^n)$  (see Corollary 7.4). This allows us to complete the proof of Proposition 3.4.

## 5. GENERALIZED COHOMOLOGY OF PRO-SPACES

In this section we expand our discussion of pro-spaces from Section 3.1. We define the 2-complete  $E$ -cohomology groups  $(E_2^\wedge)^*X$  when  $X$  is a pro-space and  $E$  is an ordinary spectrum. We then prove Proposition 3.2. We assume that the reader is familiar with the language of model structures [Hi, Ho, Q].

**5.1.  $\mathbb{Z}/2$ -cohomological homotopy theory of pro-spaces.** Recall that there is a  $\mathbb{Z}/2$ -cohomological model structure on the category of pro-spaces. The  $\mathbb{Z}/2$ -cohomological equivalences are the pro-maps that induce isomorphisms on singular  $\mathbb{Z}/2$ -cohomology. We write  $[-, -]_{\mathbb{Z}/2}^{\text{pro}}$  for sets of homotopy classes in the  $\mathbb{Z}/2$ -cohomological homotopy category  $\text{Ho}_{\mathbb{Z}/2}(\text{pro-}s\text{Set})$ .

The cofibrations of pro-spaces are easy to describe. They are the pro-maps that are isomorphic to levelwise cofibrations. In particular, every pro-space is cofibrant because every simplicial set is cofibrant.

The  $\mathbb{Z}/2$ -cohomologically fibrant objects are described by [I3, Thm. 3.3], but the complete description is complicated and more than we actually need here. We need to know that the constant pro-space  $cK(\mathbb{Z}/2, n)$  is  $\mathbb{Z}/2$ -cohomologically fibrant if  $K(\mathbb{Z}/2, n)$  is a fibrant Eilenberg-Mac Lane space [I3, Thm. 3.3]. In particular,  $c\mathbb{R}P^\infty$  is  $\mathbb{Z}/2$ -cohomologically fibrant. It follows formally that the product  $c\mathbb{R}P^\infty \times c\mathbb{R}P^\infty = c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$  is also  $\mathbb{Z}/2$ -cohomologically fibrant.

**Lemma 5.2.** *For any pro-space  $X$ ,  $[X, c\mathbb{R}P^\infty]_{\mathbb{Z}/2}^{\text{pro}}$  is naturally isomorphic to  $H^1(X; \mathbb{Z}/2)$ , and  $[X, c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)]_{\mathbb{Z}/2}^{\text{pro}}$  is naturally isomorphic to  $H^1(X; \mathbb{Z}/2) \times H^1(X; \mathbb{Z}/2)$ .*

*Proof.* We just observed that both  $c\mathbb{R}P^\infty$  and  $c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$  are  $\mathbb{Z}/2$ -cohomologically fibrant. Since  $X$  is automatically cofibrant, we know that  $[X, c\mathbb{R}P^\infty]_{\mathbb{Z}/2}^{\text{pro}}$  is equal to

$$\pi_0 \text{Map}_{\text{pro}}(X, c\mathbb{R}P^\infty) = \pi_0 \text{colim}_s \text{Map}(X_s, c\mathbb{R}P^\infty).$$

Using that  $\pi_0$  commutes with filtered colimits, this last expression can be identified with  $\text{colim}_s H^1(X_s; \mathbb{Z}/2) = H^1(X; \mathbb{Z}/2)$ .

This establishes the first statement. The proof of the second is similar.  $\square$

**Remark 5.3.** If  $X$  is an ordinary space, then the lemma implies that the maps  $[X, \mathbb{R}P^\infty] \rightarrow [cX, c\mathbb{R}P^\infty]_{\mathbb{Z}/2}^{\text{pro}}$  and  $[X, \mathbb{R}P^\infty \times \mathbb{R}P^\infty] \rightarrow [cX, c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)]_{\mathbb{Z}/2}^{\text{pro}}$  induced by the functor  $c$  are both isomorphisms.

**5.4.  $\mathbb{Z}/2$ -cohomological model structure for pro-spectra.** In addition to the  $\mathbb{Z}/2$ -cohomological model structure on  $\text{pro-}s\text{Set}$ , there is an analogous  $\mathbb{Z}/2$ -cohomological model structure on the category of pro-spectra. As for pro-spaces, the weak equivalences are the pro-maps that induce isomorphisms in singular cohomology with coefficients in  $\mathbb{Z}/2$ . The existence of the  $\mathbb{Z}/2$ -cohomological model structure on pro-spectra can be established with the general result [I3, Thm. 2.2],

where  $K$  is a set of fibrant models for the Eilenberg-Mac Lane spectra  $\Sigma^{-q}H\mathbb{Z}/2$ . We write  $[-, -]_{\mathbb{Z}/2}^{\text{pro}}$  for sets of homotopy classes in the associated  $\mathbb{Z}/2$ -cohomological homotopy category of pro-spectra.

As for pro-spaces, the cofibrations are easy to describe. They are the pro-maps that are isomorphic to levelwise cofibrations. In particular, a pro-spectrum  $X$  is cofibrant if each  $X_s$  is a cofibrant spectrum.

**Definition 5.5.** *A spectrum  $X$  is  $\mathbb{Z}/2$ -finite if the stable homotopy groups  $\pi_i X$  are all finite 2-groups and are zero for all but finitely many values of  $i$ .*

The  $\mathbb{Z}/2$ -cohomologically fibrant pro-spectra are described by a stable version of [I3, Thm. 3.3], but again the complete description is complicated and more than we actually need here. For our purposes, it suffices to observe that a pro-spectrum  $X$  is  $\mathbb{Z}/2$ -cohomologically fibrant if certain structure maps associated to the diagram  $X$  are fibrations and if each  $X_s$  is  $\mathbb{Z}/2$ -finite. The idea of the proof is to use the following lemma to simplify the notion of  $K$ -nilpotence from [I3, Defn. 3.1].

**Lemma 5.6.** *Let  $\mathcal{C}$  be the smallest class of spectra such that:*

- (1)  $\mathcal{C}$  contains the trivial spectrum  $*$ .
- (2) If  $X \rightarrow Y$  is a weak equivalence, then  $X$  belongs to  $\mathcal{C}$  if and only if  $Y$  belongs to  $\mathcal{C}$ .
- (3) If  $\Sigma^{-q}H\mathbb{Z}/2 \rightarrow X \rightarrow Y$  is a homotopy fiber sequence and  $Y$  belongs to  $\mathcal{C}$ , then so does  $X$ .

*The class  $\mathcal{C}$  is equal to the class of  $\mathbb{Z}/2$ -finite spectra.*

*Proof.* We first show that  $\mathcal{C}$  is contained in the class of  $\mathbb{Z}/2$ -finite spectra. To do this, note first that the trivial spectrum  $*$  is  $\mathbb{Z}/2$ -finite. Second, if  $X \rightarrow Y$  is a weak equivalence, then  $X$  is  $\mathbb{Z}/2$ -finite if and only if  $Y$  is  $\mathbb{Z}/2$ -finite. Third, the long exact sequence of homotopy groups shows that if  $\Sigma^{-q}H\mathbb{Z}/2 \rightarrow X \rightarrow Y$  is a homotopy fiber sequence and  $Y$  is  $\mathbb{Z}/2$ -finite, then so is  $X$ . This finishes one direction.

For the other direction, let  $X$  be a  $\mathbb{Z}/2$ -finite spectrum. We want to show that  $X$  belongs to  $\mathcal{C}$ . The proof is by induction on the order of the finite group  $\oplus \pi_i X$ . The base case is that contractible spectra belong to  $\mathcal{C}$ .

Let  $n$  be the largest number such that  $\pi_n X$  is non-zero. Choose a short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \pi_n X \rightarrow A \rightarrow 0.$$

This is always possible because  $\pi_n X$  is a finite abelian 2-group. One can construct a homotopy fiber sequence

$$\Sigma^n H\mathbb{Z}/2 \rightarrow X \rightarrow X',$$

where  $\pi_i X'$  equals  $\pi_i X$  for  $i \neq n$  and  $\pi_n X'$  equals  $A$ . Note that  $X'$  is  $\mathbb{Z}/2$ -finite, and the order of  $\oplus \pi_i X'$  is strictly smaller than the order of  $\oplus \pi_i X$ . By the induction assumption,  $X'$  belongs to  $\mathcal{C}$ . Using property (3) of the class  $\mathcal{C}$ , the fiber sequence now shows that  $X$  also belongs to  $\mathcal{C}$ .  $\square$

**Lemma 5.7.** *The functor  $c$  is a left Quillen functor from the ordinary stable model structure on spectra to the  $\mathbb{Z}/2$ -cohomological model structure on pro-spectra.*

*Proof.* By definition of pro-categories,  $c$  is left adjoint to the functor  $\lim$  that takes a pro-object  $X$  to the cofiltered limit  $\lim_s X_s$  of the diagram  $X$ . Note that  $c$

takes cofibrations to levelwise cofibrations, and  $c$  takes weak equivalences to  $\mathbb{Z}/2$ -cohomological weak equivalences because ordinary stable weak equivalences induce isomorphisms in  $\mathbb{Z}/2$ -cohomology.  $\square$

**5.8. 2-complete cohomology theories.** For any pro-spectrum  $X$  and any  $q \in \mathbb{Z}$ , the pro-spectrum  $\Sigma^q X$  is constructed by applying the functor  $\Sigma^q$  to each spectrum  $X_s$  in the diagram  $X$ .

To simplify notation, we will always assume that  $E$  is a fibrant spectrum (in any of the standard simplicial model category structures for spectra). Also, we assume that Postnikov sections  $P_n E$  have been defined functorially in such a way that the natural maps  $P_n E \rightarrow P_{n-1} E$  in the Postnikov tower of  $E$  are fibrations between fibrant spectra.

The  $\mathbb{Z}/2$ -cohomological model structure for pro-spectra allows us to construct 2-complete cohomology theories.

**Definition 5.9.** *Let  $E$  be a spectrum, and let  $X$  be a pro-spectrum. The groups  $(E_2^\wedge)^q(X)$  are defined to be  $[\Sigma^{-q} X, cE]_{\mathbb{Z}/2}^{\text{pro}}$ . For any spectrum  $X$ , the natural map  $\eta: E^q(X) \rightarrow (E_2^\wedge)^q(cX)$  is the map  $[\Sigma^{-q} X, E] \rightarrow [c\Sigma^{-q} X, cE]_{\mathbb{Z}/2}^{\text{pro}}$  induced by the functor  $c$ .*

Lemma 5.7 explains why  $\eta$  is well-defined. See [Qk] for a different approach to the same basic construction.

If  $X$  is a pro-space, we write  $(E_2^\wedge)^*(X)$  for  $(E_2^\wedge)^*(\Sigma^\infty X_+)$ . Here  $\Sigma^\infty X_+$  is constructed by applying the functor  $Z \mapsto \Sigma^\infty(Z_+)$  to each space  $X_s$  in the cofiltered diagram  $X$ .

In order to compute  $(E_2^\wedge)^*(X)$ , one needs to take a  $\mathbb{Z}/2$ -cohomologically fibrant replacement for  $cE$ . Next we consider situations in which this fibrant replacement and the map  $\eta$  are computable.

**Lemma 5.10.** *Suppose that  $E$  is a spectrum such that  $E$  is  $k$ -connected for some  $k$  and each coefficient group  $E^q$  is a finite 2-group. The Postnikov tower  $PE$  of  $E$ , viewed as a pro-spectrum, is a  $\mathbb{Z}/2$ -cohomological fibrant replacement for  $cE$ .*

*Proof.* To begin, note that the map  $cE \rightarrow PE$  induces an isomorphism in  $\mathbb{Z}/2$ -cohomology because  $E \rightarrow P_n E$  is an isomorphism in cohomology in degrees less than  $n$ . It remains only to observe that each  $P_n E$  is  $\mathbb{Z}/2$ -finite.  $\square$

Finally, we come to the main point of this section, which is to prove Proposition 3.2.

*Proof of Proposition 3.2.* For part (i), note that  $(E_2^\wedge)^*$  takes  $\mathbb{Z}/2$ -cohomological equivalences to isomorphisms because  $(E_2^\wedge)^*$  is defined in terms of the  $\mathbb{Z}/2$ -cohomological homotopy category.

For part (ii), we constructed the natural transformation  $\eta$  in Definition 5.9.

Finally, for part (iii), let  $E$  be a spectrum that is  $k$ -connected for some  $k$ . Also suppose that each coefficient group  $E^q$  is a finite 2-group. We may assume that  $\Sigma^{-q} X$  is a cofibrant spectrum. Lemma 5.10 says that the Postnikov tower  $PE$  is a  $\mathbb{Z}/2$ -cohomological fibrant replacement for  $cE$ . Then  $\text{Map}_{\text{pro}}(c\Sigma^{-q} X, PE)$  is equal to  $\text{Map}(\Sigma^{-q} X, \lim_n P_n E)$  by adjointness, which is weakly equivalent to  $\text{Map}(\Sigma^{-q} X, E)$  because  $E$  is weakly equivalent to the limit of its own Postnikov tower. Now apply  $\pi_0$  to these mapping spaces to show that  $\eta$  is an isomorphism.  $\square$

6. ÉTALE COHOMOLOGY AND  $\mathcal{SP}$ -LINEAR SCHEMES

Recall the schemes  $Q_{n-1}$  and  $DQ_n$  defined in Section 4. We wish to compute the étale  $\mathbb{Z}/2$ -cohomology rings of the schemes  $DQ_n$  and  $DQ_m \times DQ_n$  over certain base rings, and we wish to understand the behavior of these rings under certain kinds of extensions of scalars. These schemes are smooth but not proper. The technical details of this section get around the absence of properness.

In this section, we are working entirely with ideas from algebraic geometry. Homotopy theory will enter the picture again in Section 7 when we study the étale realizations of  $DQ_n$  and  $DQ_m \times DQ_n$ .

We begin with an issue of notation. If  $X$  is a scheme defined over  $\mathrm{Spec} R$  and  $R \rightarrow S$  is any ring map, then  $\mathbf{X}_S$  is the scheme  $\mathrm{Spec} S \times_{\mathrm{Spec} R} X$ . This notation will appear throughout this and the next section.

**Definition 6.1.** *Let  $S$  be a Noetherian scheme. Let  $\mathcal{SP}$  be the class of schemes that are smooth and proper over  $S$ . The class of  $\mathcal{SP}$ -linear schemes over  $S$  is the smallest class of smooth schemes over  $S$  containing  $\mathcal{SP}$  such that if  $Z \rightarrow X$  is a closed inclusion of smooth schemes with open complement  $U$  and any two of  $Z$ ,  $U$ , and  $X$  are  $\mathcal{SP}$ -linear, then so is the third.*

Recall that the linear schemes [J] are the smallest class of schemes that contain the affine spaces  $\mathbb{A}_S^n$  and are also closed under the same two-out-of-three property. This motivates the terminology of Definition 6.1. In fact,  $\mathbb{A}_S^n$  is  $\mathcal{SP}$ -linear because it is the complement  $\mathbb{P}_S^n - \mathbb{P}_S^{n-1}$ . So every linear scheme is  $\mathcal{SP}$ -linear.

**Remark 6.2.** Suppose  $T \rightarrow S$  is a map of schemes. If  $X$  is an  $\mathcal{SP}$ -linear scheme over  $S$ , then the base-change  $X \times_S T$  is an  $\mathcal{SP}$ -linear scheme over  $T$ . This follows from two facts. First, smooth, proper schemes over  $S$  pull back to smooth, proper schemes over  $T$ . Second, if  $Z \hookrightarrow X$  is a closed inclusion of schemes over  $S$  with complementary open subscheme  $U = X - Z$ , then  $Z \times_S T \rightarrow X \times_S T$  is also a closed inclusion of schemes over  $T$  with open complement  $U \times_S T$ .

**Lemma 6.3.** *The class of  $\mathcal{SP}$ -linear schemes over  $S$  is closed under finite products in the category of schemes over  $S$ .*

Recall that products in the category of schemes over  $S$  are constructed as fiber products over  $S$ .

*Proof.* Let  $P$  be any smooth, proper scheme. First we will show that  $P \times Q$  is  $\mathcal{SP}$ -linear whenever  $Q$  is  $\mathcal{SP}$ -linear. Let  $\mathcal{C}$  be the class of schemes  $Q$  such that  $P \times Q$  is  $\mathcal{SP}$ -linear. We want to show that  $\mathcal{C}$  contains all  $\mathcal{SP}$ -linear schemes. Note that  $\mathcal{C}$  contains  $\mathcal{SP}$  because the product of two smooth, proper schemes is again smooth and proper. Now consider a closed inclusion  $Z \hookrightarrow X$  of smooth schemes with open complement  $U$ . Then  $P \times Z \rightarrow P \times X$  is a closed inclusion with open complement  $P \times U$ . If any two of  $Z$ ,  $X$ , and  $U$  belong to  $\mathcal{C}$ , then two of  $P \times Z$ ,  $P \times X$ , and  $P \times U$  are  $\mathcal{SP}$ -linear. By definition, the third is also  $\mathcal{SP}$ -linear. This shows that  $\mathcal{C}$  contains the class of  $\mathcal{SP}$ -linear schemes.

Now let  $Q$  be a fixed  $\mathcal{SP}$ -linear scheme, and let  $\mathcal{D}$  be the class of schemes  $W$  such that  $W \times Q$  is  $\mathcal{SP}$ -linear. If we can show that  $\mathcal{D}$  contains all  $\mathcal{SP}$ -linear schemes, then we will be done. We have already shown that  $\mathcal{D}$  contains  $\mathcal{SP}$ .

Let  $Z \rightarrow X$  be a closed inclusion of smooth schemes with open complement  $U$ . Then  $Z \times Q \rightarrow X \times Q$  is a closed inclusion of smooth schemes with open complement

$U \times Q$ . The argument from the first paragraph of the proof applies again, and  $\mathcal{D}$  contains all  $\mathcal{SP}$ -linear schemes.  $\square$

**Example 6.4.** If  $R$  is a commutative ring in which 2 is invertible, then the scheme  $DQ_n$  is  $\mathcal{SP}$ -linear over  $\text{Spec } R$ . It is the open complement of the closed inclusion  $Q_{n-1} \hookrightarrow \mathbb{P}^n$ , and both  $Q_{n-1}$  and  $\mathbb{P}^n$  are smooth and proper. It follows that the scheme  $DQ_m \times DQ_n$  is  $\mathcal{SP}$ -linear by Lemma 6.3.

Now let  $V$  be a fixed strict Hensel local domain. This means that  $V$  is a local integral domain which satisfies Hensel's lemma and whose residue field  $k$  is separably closed. We also let  $F$  be a separably closed field containing the field of fractions of  $V$ .

**Example 6.5.** In our applications we will usually have  $k = \overline{\mathbb{F}}_p$ , the algebraic closure of the finite field  $\mathbb{F}_p$ . Also, we will have  $V = W(\overline{\mathbb{F}}_p)$ , the ring of Witt vectors as defined in [Se, Sec. II.6]. Then  $V$  is a strict Hensel local domain whose residue field is  $\overline{\mathbb{F}}_p$  by [Se, Thms. II.5.3, II.5.5, II.6.8]. Note that  $V$  and  $F$  have characteristic zero.

The following theorem generalizes well-known properties of smooth, proper schemes to  $\mathcal{SP}$ -linear schemes. It will be applied later in Proposition 7.3 to  $DQ_n$  and  $DQ_m \times DQ_n$  in order to analyze their étale realizations.

**Theorem 6.6.** *Suppose that 2 is relatively prime to  $\text{char}(k)$ , and let  $X$  be  $\mathcal{SP}$ -linear over  $\text{Spec } V$ . Then both maps in the diagram*

$$X_k \rightarrow X \leftarrow X_F$$

*induce isomorphisms in étale cohomology with coefficients in  $\mathbb{Z}/2$ .*

*Proof.* Let  $\mathcal{C}$  be the class of schemes  $X$  over  $\text{Spec } V$  such that both maps induce étale cohomology isomorphisms. We want to show that  $\mathcal{C}$  contains all  $\mathcal{SP}$ -linear schemes.

If  $X$  is proper over  $\text{Spec } V$ , then  $H_{\text{et}}^*(X; \mathbb{Z}/2) \rightarrow H_{\text{et}}^*(X_k; \mathbb{Z}/2)$  is an isomorphism by [M, Cor. VI.2.7] or [SGA4, Cor. XII.5.5]. If  $X$  is both smooth and proper then  $H_{\text{et}}^*(X; \mathbb{Z}/2) \rightarrow H_{\text{et}}^*(X_F; \mathbb{Z}/2)$  is an isomorphism by [M, proof of Cor. VI.4.2] or [SGA4, Cor. XVI.2.2]. Therefore  $X$  belongs to  $\mathcal{C}$ , and  $\mathcal{C}$  contains  $\mathcal{SP}$ .

Now suppose that  $Z \hookrightarrow X$  is a closed inclusion of smooth  $V$ -schemes with open complement  $U$ . We have a diagram

$$\begin{array}{ccccccc} \cdots & \longleftarrow & H_{\text{et}}^p(U_F; \mathbb{Z}/2) & \longleftarrow & H_{\text{et}}^p(X_F; \mathbb{Z}/2) & \longleftarrow & H_{\text{et}}^{p-2c}(Z_F; \mathbb{Z}/2) \longleftarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longleftarrow & H_{\text{et}}^p(U; \mathbb{Z}/2) & \longleftarrow & H_{\text{et}}^p(X; \mathbb{Z}/2) & \longleftarrow & H_{\text{et}}^{p-2c}(Z; \mathbb{Z}/2) \longleftarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longleftarrow & H_{\text{et}}^p(U_k; \mathbb{Z}/2) & \longleftarrow & H_{\text{et}}^p(X_k; \mathbb{Z}/2) & \longleftarrow & H_{\text{et}}^{p-2c}(Z_k; \mathbb{Z}/2) \longleftarrow \cdots \end{array}$$

in which the rows are long exact Gysin sequences (for the middle row, see the remark following this proof). Here  $c$  is the codimension of  $Z$  in  $X$ . If any two of  $Z$ ,  $X$ , and  $U$  belong to  $\mathcal{C}$ , then both maps in two of the columns are isomorphisms. By the five lemma, both maps in the third column are also isomorphisms, so the third scheme belongs to  $\mathcal{C}$ . This shows that  $\mathcal{C}$  contains all  $\mathcal{SP}$ -linear schemes.  $\square$



**Remark 6.7.** Beware that the middle row in the above diagram is a Gysin sequence over a base scheme  $\text{Spec } V$  in which  $V$  is not a field. By [M, Thm. VI.5.1] (or [SGA4, XVI.3.7]), there is a long exact sequence

$$\cdots \leftarrow H_{\text{et}}^p(U; \mathbb{Z}/2) \leftarrow H_{\text{et}}^p(X; \mathbb{Z}/2) \leftarrow H_{\text{et}}^{p-2c}(Z; \mathcal{F}) \leftarrow \cdots,$$

where  $\mathcal{F}$  is a sheaf on  $Z$  that is locally isomorphic to the constant sheaf  $\mathbb{Z}/2$ . Since  $\text{Aut}(\mathbb{Z}/2)$  is trivial, it follows that  $\mathcal{F}$  must be the constant sheaf  $\mathbb{Z}/2$ .

This issue explains why the statement of Theorem 6.6 is specialized to coefficients in  $\mathbb{Z}/2$ , rather than the usual twisted coefficients  $\mu_l(q)$  of  $l$ th roots of unity with  $l$  relatively prime to  $\text{char}(k)$ .

**Example 6.8.** Consider the scheme  $DQ_n$  defined over  $\overline{\mathbb{F}}_p$ , the algebraic closure of the finite field  $\mathbb{F}_p$  with  $p$  a prime greater than 2. Recall the ring of Witt vectors  $V = W(\overline{\mathbb{F}}_p)$  from Example 6.5. Note that  $DQ_n$  is defined over  $V$  since it is in fact defined over  $\mathbb{Z}$ .

Let  $F$  be the separable closure of the field of fractions of  $V$ . Theorem 6.6 implies that  $H_{\text{et}}^*(DQ_n; \mathbb{Z}/2)$  is isomorphic to  $H_{\text{et}}^*((DQ_n)_F; \mathbb{Z}/2)$ . But  $F$  has characteristic zero, so [M, Thm. III.3.12] together with [M, Cor. VI.4.3] (or [SGA4, Thm. XI.4.4] together with [SGA4, Cor. XVI.1.6]) tells us that  $H_{\text{et}}^*((DQ_n)_F; \mathbb{Z}/2)$  is isomorphic to the singular cohomology ring  $H^*(DQ_n(\mathbb{C}); \mathbb{Z}/2)$ . Since  $DQ_n(\mathbb{C})$  is homotopy equivalent to  $\mathbb{R}P^n$  by [Lw, Lem. 6.3], we have computed that

$$H_{\text{et}}^*(DQ_n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1}),$$

where  $x$  is a class of degree 1.

The same computation works for  $DQ_{n/k}$  where  $k$  is any separably closed field of characteristic different from 2. Use [M, Cor. VI.4.3] or [SGA4, Cor. XVI.1.6] to reduce to the cases  $k = \overline{\mathbb{F}}_p$  or  $F = \mathbb{C}$ , depending on whether the characteristic is positive or zero, similarly to the proof of Corollary 7.4 below.

**Example 6.9.** Consider the scheme  $DQ_m \times DQ_n$  over a separably closed field  $k$  such that  $\text{char}(k) \neq 2$ . As in Example 6.8, we compute that  $H_{\text{et}}^*(DQ_m \times DQ_n; \mathbb{Z}/2)$  is isomorphic to

$$H^*(\mathbb{R}P^m \times \mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2]/(x_1^{m+1}, x_2^{n+1}),$$

where  $x_1$  and  $x_2$  are classes of degree 1.

## 7. ÉTALE REALIZATIONS

Recall that to any scheme  $X$  one can associate a pro-simplicial set  $\text{Et}(X)$  called the **étale realization** of  $X$ . This is defined in [F, Def. 4.4], where it is called the ‘étale topological type’. The construction of  $\text{Et}(X)$  is technical, but fortunately the details are not necessary for the arguments below.

Note that if  $X$  is a scheme over  $\text{Spec } R$  and  $R \rightarrow T$  is a ring map, then the base-change map  $X_T \rightarrow X$  induces a map  $\text{Et}(X_T) \rightarrow \text{Et}(X)$ .

Here are the facts that we will need about étale realizations. They are analogues of the results about étale cohomology used in Section 6.

- (1) [F, Prop. 5.9]: If  $X$  is any scheme, then  $H_{\text{et}}^*(X; M)$  is naturally isomorphic to  $H^*(\text{Et}(X); M)$  for any constant coefficients  $M$ . In fact, a stronger statement concerning twisted coefficients can be made, but we won’t need it.

- (2) [F, Thm. 8.4] or [AM, Thm. 12.9]: If  $X$  is a scheme of finite type over  $\mathbb{C}$ , then there is a canonical zig-zag in  $\text{pro-}\mathcal{S}\text{et}$  of the form

$$\text{Et}(X) \leftarrow \text{SEt}(X) \rightarrow cX(\mathbb{C})$$

where both maps are  $\mathbb{Z}/l$ -cohomological equivalences, for any prime  $l$ . Here  $\text{SEt}(-)$  is a certain functor defined in [F, Thm. 8.4].

- (3) [F, Prop. 8.6, Prop. 8.7] or [AM, Cor. 12.13]: Suppose that  $V$  is a strict Hensel local domain—that is, a local domain satisfying Hensel’s lemma whose residue field  $k$  is separably closed. Let  $F$  be a separably closed field containing the quotient field of  $V$ . If  $X$  is a scheme that is both smooth and proper over  $\text{Spec } V$ , then the maps of pro-spaces

$$\text{Et}(X_k) \rightarrow \text{Et}(X) \leftarrow \text{Et}(X_F)$$

are both  $\mathbb{Z}/l$ -cohomological equivalences, for any prime  $l$  different from  $\text{char}(k)$ .

We need one additional property of étale realizations, concerning their behavior with respect to field extensions.

**Lemma 7.1.** *Let  $E \hookrightarrow F$  be an inclusion of separably closed fields. If  $X$  is a scheme over  $E$ , then the canonical map  $\text{Et}(X_F) \rightarrow \text{Et}(X)$  is a  $\mathbb{Z}/l$ -cohomological equivalence for any prime  $l$  different from  $\text{char}(F)$ .*

*Proof.* By the definition of  $\mathbb{Z}/l$ -cohomological equivalences and property (1), we need only observe that the map  $H_{\text{et}}^*(X; \mathbb{Z}/l) \rightarrow H_{\text{et}}^*(X_F; \mathbb{Z}/l)$  is an isomorphism by [M, Cor. VI.4.3] or [SGA4, Cor. XVI.1.6].  $\square$

The careful reader will notice that this lemma is slightly different than [AM, Cor. 12.12]. We have not assumed that  $X$  is proper; instead, we have assumed that  $l$  is relatively prime to the characteristics of the fields.

We wish to compute  $\text{Et}(DQ_n)$  and  $\text{Et}(DQ_m \times DQ_n)$  and to show that they can be connected by  $\mathbb{Z}/2$ -cohomological equivalences to the constant pro-spaces  $c\mathbb{R}P^n$  and  $c(\mathbb{R}P^m \times \mathbb{R}P^n)$ . This will go a long way towards establishing Proposition 3.4. As the scheme  $DQ_n$  can be lifted to characteristic zero, and we know that  $DQ_n(\mathbb{C})$  is homotopy equivalent to  $\mathbb{R}P^n$  by [Lw, Lem. 6.3], properties (1)–(3) and Lemma 7.1 can almost be used to provide the required equivalences. Although  $DQ_n$  is smooth, it is not proper; thus property (3) does not apply directly. However, since  $DQ_n$  is  $\mathcal{SP}$ -linear, the results of Section 6 will get around this problem.

Now assume that there is a diagram

$$(7.2) \quad \begin{array}{ccccc} A & \longrightarrow & V & \twoheadrightarrow & E \\ \downarrow & & & & \downarrow \\ \mathbb{C} & \twoheadrightarrow & & & K \end{array}$$

of rings in which the indicated maps are inclusions and where

- $V$  is a strict Hensel local domain with separably closed residue field  $k$ ,
- $E$  and  $K$  are separably closed fields (and so  $E$  is an extension of the quotient field of  $V$ ).

If  $X$  is a scheme over  $A$ , one obtains schemes  $X_V$ ,  $X_E$ ,  $X_{\mathbb{C}}$ ,  $X_K$ , and  $X_k$  via base change. Let  $X_{\mathbb{C}}(\mathbb{C})$  denote the topological space of  $\mathbb{C}$ -valued points of  $X_{\mathbb{C}}$ , equipped with the analytic topology.

**Proposition 7.3.** *Suppose given a diagram as in (7.2) such that the residue field  $k$  of  $V$  has characteristic different from 2. For any scheme  $X$  over  $A$ , there is a natural diagram*

$$\mathrm{Et}(X_k) \rightarrow \mathrm{Et}(X_V) \leftarrow \mathrm{Et}(X_E) \leftarrow \mathrm{Et}(X_K) \rightarrow \mathrm{Et}(X_{\mathbb{C}}) \leftarrow \mathrm{SEt}(X_{\mathbb{C}}) \rightarrow cX_{\mathbb{C}}(\mathbb{C})$$

*of pro-spaces. If  $X$  is  $\mathcal{SP}$ -linear, then each map is a  $\mathbb{Z}/2$ -cohomological weak equivalence. Here  $\mathrm{SEt}$  is a functor from schemes over  $\mathbb{C}$  to pro-spaces defined in [F, Thm. 8.4].*

Recall the  $\mathbb{Z}/2$ -cohomological homotopy category  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}\mathcal{S}\mathrm{et})$  of pro-spaces that is briefly described in Section 3 and is discussed more carefully in Section 5. The point of the proposition is that when  $X$  is  $\mathcal{SP}$ -linear there is a natural isomorphism between  $cX_{\mathbb{C}}(\mathbb{C})$  and  $\mathrm{Et}(X_k)$  in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}\mathcal{S}\mathrm{et})$ .

*Proof.* The first two maps are  $\mathbb{Z}/2$ -cohomological weak equivalences as a result of Theorem 6.6 and property (1). The middle two maps  $\mathrm{Et}(X_K) \rightarrow \mathrm{Et}(X_E)$  and  $\mathrm{Et}(X_K) \rightarrow \mathrm{Et}(X_{\mathbb{C}})$  are  $\mathbb{Z}/2$ -cohomological equivalences by Lemma 7.1. Finally, the last two maps are  $\mathbb{Z}/2$ -cohomological weak equivalences because of [F, Thm. 8.4].  $\square$

**Corollary 7.4.** *Let  $k$  be a separably closed field of characteristic not equal to 2, and consider the schemes  $DQ_n$  and  $DQ_m \times DQ_n$  defined over  $k$ . In  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}\mathcal{S}\mathrm{et})$ , there are isomorphisms*

$$c\mathbb{R}P^n \rightarrow \mathrm{Et}(DQ_n)$$

*and*

$$c(\mathbb{R}P^m \times \mathbb{R}P^n) \rightarrow \mathrm{Et}(DQ_m \times DQ_n).$$

*Proof.* First suppose that  $k$  has characteristic zero. Choose an embedding  $\bar{\mathbb{Q}} \hookrightarrow k$ , and note that  $DQ_{n/k}$  is the base extension of  $DQ_n/\bar{\mathbb{Q}}$ . There is a zig-zag of maps

$$\mathrm{Et}(DQ_{n/\mathbb{C}}) \rightarrow \mathrm{Et}(DQ_{n/\bar{\mathbb{Q}}}) \leftarrow \mathrm{Et}(DQ_{n/k})$$

which are both  $\mathbb{Z}/2$ -cohomological equivalences by Lemma 7.1. Finally, property (2) from the beginning of this section says that  $\mathrm{Et}(DQ_{n/\mathbb{C}})$  is  $\mathbb{Z}/2$ -cohomologically equivalent to  $c(\mathbb{R}P^n)$ , since  $DQ_n(\mathbb{C})$  is homotopy equivalent to  $\mathbb{R}P^n$  by [Lw, Lem. 6.3]. The same argument works for  $DQ_m \times DQ_n$ .

Now suppose that  $k$  has characteristic  $p$ , where  $p > 2$ . Choose an embedding  $\mathbb{F}_p^s \hookrightarrow k$  where  $\mathbb{F}_p^s$  is the separable closure of the finite field  $\mathbb{F}_p$ . Note that  $DQ_{n/k}$  is the base extension of  $DQ_n/\mathbb{F}_p^s$ , so Lemma 7.1 gives us a  $\mathbb{Z}/2$ -cohomological equivalence

$$\mathrm{Et}(DQ_{n/k}) \rightarrow \mathrm{Et}(DQ_{n/\mathbb{F}_p^s}).$$

It therefore suffices to consider the case  $k = \mathbb{F}_p^s$ .

Next consider the inclusion  $\mathbb{F}_p^s \hookrightarrow \bar{\mathbb{F}}_p$ , where  $\bar{\mathbb{F}}_p$  is the algebraic closure of  $\mathbb{F}_p$ . This is an inclusion of separably closed fields, so Lemma 7.1 again gives us a  $\mathbb{Z}/2$ -cohomological equivalence

$$\mathrm{Et}(DQ_{n/\bar{\mathbb{F}}_p}) \rightarrow \mathrm{Et}(DQ_{n/\mathbb{F}_p^s}).$$

So it further suffices to consider the case  $k = \bar{\mathbb{F}}_p$ .

Recall from Example 6.5 that  $V = W(\bar{\mathbb{F}}_p)$  is the strict Hensel local domain of Witt vectors for  $\bar{\mathbb{F}}_p$ . Let  $A = \mathbb{Z}$ , and let  $E$  be a separable closure of the quotient field

of  $V$ . For  $K$  we choose any field containing both  $E$  and  $\mathbb{C}$ , such as the compositum of  $E$  and  $\mathbb{C}$ .

We are now in a position to apply the preceding proposition, since  $DQ_n$  is  $\mathcal{SP}$ -linear. We find that  $\mathrm{Et}(DQ_n/\overline{\mathbb{F}}_p)$  is isomorphic to  $cDQ_n/\mathbb{C}(\mathbb{C})$  in the  $\mathbb{Z}/2$ -cohomological homotopy category. We have already observed many times that the spaces  $DQ_n/\mathbb{C}(\mathbb{C})$  and  $\mathbb{R}P^n$  are homotopy equivalent, so this completes the proof for  $DQ_n$ .

Everything works the same for the product  $DQ_m \times DQ_n$ , using that this scheme is  $\mathcal{SP}$ -linear by Lemma 6.3.  $\square$

**Remark 7.5.** We will need the following fact later in the proof of Proposition 3.4 given at the end of this section. Consider the rational point  $* = [1 : 0 : 0 : \dots : 0]$  of the scheme  $DQ_n$  defined over a field  $k$ . We regard  $*$  as a map of schemes  $\mathrm{Spec} k \rightarrow DQ_n$ . There is an induced map  $j_1 = \mathrm{id} \times * : DQ_m \rightarrow DQ_m \times DQ_n$ . We then have a diagram in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro-sSet})$  of the form

$$\begin{array}{ccc} c(\mathbb{R}P^m) & \xrightarrow{\cong} & \mathrm{Et}(DQ_m) \\ \downarrow & & \downarrow \mathrm{Et}(j_1) \\ c(\mathbb{R}P^m \times \mathbb{R}P^n) & \xrightarrow{\cong} & \mathrm{Et}(DQ_m \times DQ_n) \end{array}$$

where the two isomorphisms are from Corollary 7.4 and the left vertical map comes from the evident inclusion  $\mathbb{R}P^m \hookrightarrow \mathbb{R}P^m \times \mathbb{R}P^n$  (using the point  $[1 : 0 : \dots : 0]$  of  $\mathbb{R}P^n$ ). This diagram is commutative because of the naturality of the maps in Proposition 7.3.

The same observations apply to the evident map  $j_2 : DQ_n \rightarrow DQ_m \times DQ_n$ .

We are now ready to complete the last remaining detail of the paper.

*Proof of Proposition 3.4.* Assume that a sums-of-squares formula of type  $[r, s, n]$  exists over a field  $k$  with  $\mathrm{char}(k) \neq 2$ . Let  $k^s$  be the separable closure of  $k$ , and note that the sums-of-squares formula over  $k$  automatically exists over  $k^s$  also. Hence we may assume that  $k$  is separably closed.

As explained in [DI1, Sec. 1], the sums-of-squares formula induces a map  $f : DQ_{r-1} \times DQ_{s-1} \rightarrow DQ_{n-1}$ . At the level of points, this map sends the pair  $([x_1 : \dots : x_r], [y_1 : \dots : y_s])$  to  $[z_1 : \dots : z_n]$ , using the notation from Section 1.

Let  $* = [1 : 0 : 0 : \dots : 0]$  be the basepoint in both  $DQ_{r-1}$  and  $DQ_{s-1}$ . Let  $j_1 : DQ_{r-1} \hookrightarrow DQ_{r-1} \times DQ_{s-1}$  be the closed inclusion as defined in Remark 7.5. Define  $j_2$  similarly.

Recall from Example 6.8 that the group  $H_{\mathrm{et}}^1(DQ_{n-1}; \mathbb{Z}/2)$  is isomorphic to  $\mathbb{Z}/2$ ; let  $x$  be the generator. Let  $x_1$  and  $x_2$  be the classes  $\pi_1^*(x)$  and  $\pi_2^*(x)$  of  $H_{\mathrm{et}}^1(DQ_{r-1} \times DQ_{s-1}; \mathbb{Z}/2)$ , where  $\pi_1$  and  $\pi_2$  are the obvious projections. Note that  $j_1^*(x_1) = x = j_2^*(x_2)$  and  $j_2^*(x_1) = 0 = j_1^*(x_2)$ . This shows that  $x_1$  and  $x_2$  are distinct non-zero classes. But by Example 6.9,  $H_{\mathrm{et}}^1(DQ_{r-1} \times DQ_{s-1}; \mathbb{Z}/2)$  is isomorphic to  $\mathbb{Z}/2 \times \mathbb{Z}/2$ , so  $x_1$  and  $x_2$  are generators for this group.

Since the sums-of-squares formula is bilinear, the compositions  $fj_1 : DQ_{r-1} \rightarrow DQ_{n-1}$  and  $fj_2 : DQ_{s-1} \rightarrow DQ_{n-1}$  are linear inclusions. This shows that  $f^*(x) = x_1 + x_2$ . (compare [DI1, Prop. 2.5], which is the analogous result for motivic cohomology).

According to Lemma 5.2, for any scheme  $X$ , the set  $[\mathrm{Et}(X), c\mathbb{R}P^\infty]_{\mathbb{Z}/2}^{\mathrm{pro}}$  of homotopy classes in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}s\mathrm{Set})$  is naturally isomorphic to  $H^1(\mathrm{Et}(X); \mathbb{Z}/2)$ , which in turn is isomorphic to  $H_{\mathrm{et}}^1(X; \mathbb{Z}/2)$  by property (1) from the beginning of Section 7. The element  $x$  of  $H_{\mathrm{et}}^1(DQ_{n-1}; \mathbb{Z}/2)$  therefore gives us a homotopy class  $\mathrm{Et}(DQ_{n-1}) \rightarrow c\mathbb{R}P^\infty$  in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}s\mathrm{Set})$ . Similarly, the elements  $x_1$  and  $x_2$  of  $H_{\mathrm{et}}^1(DQ_{r-1} \times DQ_{s-1}; \mathbb{Z}/2)$  give us a homotopy class  $\mathrm{Et}(DQ_{r-1} \times DQ_{s-1}) \rightarrow c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$  in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}s\mathrm{Set})$ ; here we are using the isomorphism between

$$H^1(DQ_{r-1} \times DQ_{s-1}; \mathbb{Z}/2) \times H^1(DQ_{r-1} \times DQ_{s-1}; \mathbb{Z}/2)$$

and

$$[\mathrm{Et}(DQ_{r-1} \times DQ_{s-1}), c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)]_{\mathbb{Z}/2}^{\mathrm{pro}}$$

from the second part of Lemma 5.2.

We now have a diagram

$$\begin{array}{ccc} \mathrm{Et}(DQ_{r-1} \times DQ_{s-1}) & \longrightarrow & c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) \\ \mathrm{Et}(f) \downarrow & & \downarrow c(\mu) \\ \mathrm{Et}(DQ_{n-1}) & \longrightarrow & c\mathbb{R}P^\infty \end{array}$$

in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}s\mathrm{Set})$ . On cohomology, the left vertical map takes  $x$  to  $x_1 + x_2$ . Commutativity of this diagram follows from this and from the fact that  $\mu$  represents the element  $y_1 + y_2$  of  $H^1(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}/2)$ , where  $y$  is the generator of  $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$  and  $y_1$  and  $y_2$  are the classes  $\pi_1^*(y)$  and  $\pi_2^*(y)$  in  $H^1(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}/2)$ .

Using the maps supplied by Corollary 7.4, we obtain a commutative diagram

$$\begin{array}{ccccc} c(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}) & \xrightarrow{\sim} & \mathrm{Et}(DQ_{r-1} \times DQ_{s-1}) & \longrightarrow & c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) \\ \downarrow & & \downarrow & & \downarrow \mu \\ c\mathbb{R}P^{n-1} & \xrightarrow{\sim} & \mathrm{Et}(DQ_{n-1}) & \longrightarrow & c\mathbb{R}P^\infty \end{array}$$

in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}s\mathrm{Set})$ , where the left vertical map is defined using the homotopy-inverse of the map  $c\mathbb{R}P^{n-1} \rightarrow \mathrm{Et}(DQ_{n-1})$ . The outer square in this diagram is the desired commutative square. We still need to show that the horizontal maps correspond to the canonical inclusions; this identification is necessary for the arguments in Section 3.

For the bottom composite, there are only two homotopy classes  $c\mathbb{R}P^{n-1} \rightarrow c\mathbb{R}P^\infty$ , and they are classified by the map that they induce on  $H^1(-; \mathbb{Z}/2)$  (using Lemma 5.2). We have chosen  $\mathrm{Et}(DQ_{n-1}) \rightarrow c\mathbb{R}P^\infty$  so that the bottom composite is the non-zero homotopy class. On the other hand, any linear inclusion  $\mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^\infty$  also represents the non-zero homotopy class.

For the top composite, there are exactly sixteen homotopy classes of maps  $h: c(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1}) \rightarrow c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$ , and again they are classified by the induced map on  $H^1(-; \mathbb{Z}/2)$ . More concretely, such maps are classified by the values of the four expressions

$$j_1^* h^* \pi_1^*(y), \quad j_1^* h^* \pi_2^*(y), \quad j_2^* h^* \pi_1^*(y), \quad \text{and} \quad j_2^* h^* \pi_2^*(y)$$

(each of which has exactly two possible values). Here  $j_1$ ,  $j_2$ ,  $\pi_1$ , and  $\pi_2$  are the evident inclusions and projections, as above.

Let  $h$  denote the top composite in the above diagram. Check that  $j_1^* h^* \pi_1^*(y)$  and  $j_2^* h^* \pi_2^*(y)$  are non-zero, while  $j_1^* h^* \pi_2^*(y)$  and  $j_2^* h^* \pi_1^*(y)$  are zero; this step uses the commutativity of the square in Remark 7.5. Finally, if

$$j \times j: \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \rightarrow \mathbb{R}P^\infty \times \mathbb{R}P^\infty$$

is the standard inclusion, observe that  $j_1^*(j \times j)^* \pi_1^*(y)$  and  $j_2^*(j \times j)^* \pi_2^*(y)$  are non-zero, while  $j_1^*(j \times j)^* \pi_2^*(y)$  and  $j_2^*(j \times j)^* \pi_1^*(y)$  are zero. By the observation in the previous paragraph, it follows that  $h$  and  $j \times j$  represent the same map in  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}sSet)$ . This completes the proof.  $\square$

**Remark 7.6.** There is another, more elegant method for proving Proposition 3.4, but we have avoided it because it requires more machinery. The idea is to prove a motivic version of the proposition. Let  $k$  be a field, and recall that Morel and Voevodsky [MV] have constructed a model category representing the motivic homotopy theory of smooth schemes over  $k$ .

If  $\mathrm{char}(k) \neq 2$  and a sums-of-squares formula of type  $[r, s, n]$  exists over  $k$ , then there exists a diagram

$$\begin{array}{ccc} DQ_{r-1} \times DQ_{s-1} & \xrightarrow{j \times j} & DQ_\infty \times DQ_\infty \\ \downarrow & & \downarrow \\ DQ_{n-1} & \xrightarrow{j} & DQ_\infty \end{array}$$

in the motivic homotopy category, where the horizontal maps are standard inclusions and  $DQ_\infty$  is the colimit (or ascending union) of the schemes  $DQ_n$  for all  $n$ .

The paper [I2] constructs a functor  $\mathrm{Et}$  from the motivic homotopy category to  $\mathrm{Ho}_{\mathbb{Z}/2}(\mathrm{pro}\text{-}sSet)$ . This functor extends the usual étale realization on schemes, and applying it to the above diagram yields a diagram of pro-spaces. Some work is required to identify  $\mathrm{Et}(DQ_\infty)$  with  $c\mathbb{R}P^\infty$  and  $\mathrm{Et}(DQ_\infty \times DQ_\infty)$  with  $c(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$  (and the map between them with  $\mu$ ), but this then yields the desired diagram of Proposition 3.4.

Another relevant tool for this approach is the stable étale realization of [Qk] from the motivic stable homotopy category to another stable homotopy category that is closely related to the  $\mathbb{Z}/2$ -cohomological homotopy category of pro-spectra.

## REFERENCES

- [Ad] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, Univ. Chicago Press, 1974.
- [An] V. Angeltveit,  *$A_\infty$  obstruction theory and the strict associativity of  $E/I$* , preprint, 2005.
- [AM] M. Artin and B. Mazur, *Étale Homotopy*, Lecture Notes in Mathematics **100**, Springer, 1969.
- [As] L. Astey, *Geometric dimension of bundles over real projective spaces*, Quart. J. Math. Oxford (2) **31** (1980), 139–155.
- [A] M. F. Atiyah, *Immersion and embeddings of manifolds*, Topology **1** (1962), 125–132.
- [AH] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, 1961 Proc. Sympos. Pure Math., Vol. III, 7–38, Amer. Math. Soc., 1961.
- [B] N. A. Baas, *On bordism theory of manifolds with singularities*, Math. Scand. **33** (1973), 279–302.
- [BDM] R. R. Bruner, D. M. Davis, and M. Mahowald, *Nonimmersions of real projective spaces implied by  $tmf$* , in Recent progress in homotopy theory (Baltimore, 2000), 45–68, Contemp. Math. **293**, Amer. Math. Soc., 2002.

- [D1] D. M. Davis, *Generalized homology and the generalized vector field problem*, Quart. J. Math. Oxford **25** (1974), 169–193.
- [D2] D. M. Davis, *A strong nonimmersion theorem for real projective spaces*, Ann. Math. **120** (1984), 517–528.
- [DI1] D. Dugger and D. C. Isaksen, *The Hopf condition for bilinear forms over arbitrary fields*, Ann. Math., to appear.
- [DI2] D. Dugger and D. C. Isaksen, *Algebraic K-theory and sums-of-squares formulas*, Doc. Math. **10** (2005), 357–366.
- [EKMM] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole*, Mathematical Surveys and Monographs **47**, Amer. Math. Soc., 1997.
- [F] E. M. Friedlander, *Etale Homotopy of Simplicial Schemes*, Ann. of Math. Stud. **104**, Princeton Univ. Press, 1982.
- [G] P. G. Goerss, *Associative MU-algebras*, preprint.
- [Hi] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs **99**, Amer. Math. Soc., 2003.
- [H] H. Hopf, *Ein topologischer Beitrag zur reellen Algebra*, Comment. Math. Helv. **13** (1940/41), 219–239.
- [Ho] M. Hovey, *Model categories*, Mathematical Surveys and Monographs **63**, Amer. Math. Soc., 1999.
- [HSS] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000) 149–208.
- [H1] A. Hurwitz, *Über die Komposition der quadratischen Formen von beliebig vielen Variabeln*, Nach. V. der. Ges. der Wiss., Göttingen, Math. Phys. kl. (1898) 309–316. Reprinted in *Math. Werke*, Band 2, 565–571, Birkhäuser, 1963.
- [H2] A. Hurwitz, *Über die Komposition der quadratischen Formen*, Math. Ann. **88** (1923) 1–25. Reprinted in *Math. Werke*, Band 2, 641–666, Birkhäuser, 1963.
- [I1] D. C. Isaksen, *A model structure on the category of pro-simplicial sets*, Trans. Amer. Math. Soc. **353** (2001), 2805–2841.
- [I2] D. C. Isaksen, *Étale realization on the  $\mathbb{A}^1$ -homotopy theory of schemes*, Adv. Math. **184** (2004), 37–63.
- [I3] D. C. Isaksen, *Completions of pro-spaces*, Math. Z. **250** (2005), 113–143.
- [J] U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Mathematics **1400**, Springer, 1990.
- [L] K. Y. Lam, *Topological methods for studying the composition of quadratic forms*, Quadratic and Hermitian Forms, (Hamilton, Ont., 1983), 173–192, Canadian Mathematical Society Conference Proceedings **4**, Amer. Math. Soc., 1984.
- [Lw] P. S. Landweber, *Fixed point free conjugations on complex manifolds*, Ann. Math. (2) **86** (1967), 491–502.
- [M] J. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton Univ. Press, 1980.
- [Mo] F. Morel, *Ensembles profinis simpliciaux et interprétation géométrique du foncteur  $T$* , Bull. Soc. Math. France **124** (1996), 347–373.
- [MV] F. Morel and V. Voevodsky,  *$\mathbb{A}^1$ -homotopy theory*, Inst. Hautes Etudes Sci. Publ. Math. **90** (2001), 45–143.
- [O] S. Oka, *Multiplications on the Moore spectrum*, Mem. Fac. Sci. Kyushu Univ. Ser. A **38** (1984), no. 2, 257–276.
- [Qk] G. Quick, *Stable étale realization and étale cobordism*, arXiv:math.AG/0608313, preprint, 2006.
- [Q] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics **43**, Springer, 1967.
- [R] J. Radon, *Lineare Scharen orthogonaler Matrizen*, Abh. Math. Sem. Univ. Hamburg **1** (1922) 1–14.
- [Se] J. P. Serre, *Local fields*, Graduate Texts in Math. **67**, Springer-Verlag New York, 1979.
- [SGA4] *Théorie des topos et cohomologie étale des schémas*, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), by M. Artin, A. Grothendieck, and J. L. Verdier, Lecture Notes in Mathematics **269**, **270**, and **305**, Springer, 1972.
- [Sh] D. B. Shapiro, *Products of sums of squares*, Expo. Math. **2** (1984), 235–261.

- [SS] D. B. Shapiro and M. Szyjewski, *Product formulas for quadratic forms*, Bol. Soc. Mat. Mexicana **37** (1992), 463–474.
- [St] E. Stiefel, *Über Richtungsfelder in den projektiven Räumen und einen Satz aus der reellen Algebra*, Comment. Math. Helv. **13** (1940/41), 201–218.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MI 48202

*E-mail address:* `ddugger@math.uoregon.edu`

*E-mail address:* `isaksen@math.wayne.edu`